Interpretable reparameterizations of growth curve models

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June 18, 2010

Motivation

Clinical: Children's affiliation with delinquent peers

- Researchers may be interested in detecting when association with delinquent peers is most uniform across children (Stoolmiller, 1994).

Public Health: Chemical absorption and elimination

- Researchers may be interested in describing individual differences in the rate of plasma phosphate depletion and rebound following carbohydrate ingestion (Obeid, Dimachkie, & Hlais, 2010; Zerbe, 1979).

Sociology: Infant growth and malnutrition in developing countries

- Researchers may be interested in predicting individual differences in growth at multiple points on curves which are not readily described by polynomials (UNICEF, 1997, 2008).

Motivation

It is not straightforward to accomplish any of these tasks in an SEM framework.

Clinical: Children's affiliation with delinquent peers
- **Known:** How to compute the point of most similarity in delinquent affiliation after a model has already been estimated.
- **Unknown:** How to directly estimate this point, or assess whether it is moderated.

Public Health: Chemical absorption and elimination
- **Known:** Finding individually varying inflection points using multilevel modeling.
- **Unknown:** Finding individually varying inflection points using SEM.

Sociology: Infant growth and malnutrition in developing countries
- **Known:** How to recode intercept to assess/predict individual differences at varying points on polynomial curves.
- **Unknown:** How to recode the intercept to assess/predict individual differences at varying points on more complex nonlinear curves.

Reparameterization

Each of these substantively relevant research questions can be addressed by *reparameterizing* available latent growth curve models (LGMs) to allow us to estimate parameters to which we do not usually have access.

For example…

Clinical: Children's affiliation with delinquent peers
- Reparameterize a linear LGM so that the point of greatest similarity is a model parameter.

Public Health: Chemical absorption and elimination
- Reparameterize a piecewise LGM so that the transition point between phases of phosphate absorption and release is a random effect.

Sociology: Infant growth and malnutrition in developing countries
- Reparameterize a LGM with an exponential component so that predicted infant weight at any desired occasion is a random effect.

Reparameterization

In the context of longitudinal data analysis, reparameterization assumes that:

- we have selected a model appropriate for the domain of study.
- we wish to quantify some aspect of growth as a model parameter that is *not already represented* in the function.

Reparameterizations of a model are statistically equivalent. This implies:

- Reparameterized models have identical fit.
- Reparameterized models have the same number of free parameters.

Reparameterization

Our three motivating examples illustrate substantive reasons we might want to reparameterize available models. Stated more generally, these reasons include:

- More convenient to directly estimate a parameter than to compute post hoc; we can directly obtain SE and CI for the parameter (as in delinquent peer affiliation example).
- Desirable to investigate whether the aspect of change is moderated or predicted by other variables (as in infant growth example).
- Desirable to have the option of treating a parameter as a known value, an estimated fixed coefficient, or a random coefficient (as in the public health and infant growth studies).

Reparameterization

In the methodology literature there is a history of reparameterizing conventional models to aid in addressing specific substantive questions (e.g., Harring, Cudeck, & du Toit, 2006).

For example…

- Choi, Harring, and Hancock (2009) reparameterize a logistic model to estimate lower and upper asymptotes, surge points, and jerk points (in SEM).
- Cudeck and du Toit (2002) reparameterize a quadratic curve to estimate when and where it attains its maximum/minimum (in MLM).
- Rausch (2004, 2008) reparameterizes the negative exponential curve to estimate a "half-life" parameter (in MLM).

Yet, reparameterization is applied very little outside of the methodological literature.

Reparameterization

The three motivating examples suggest that often substantive areas could benefit from reparameterization to estimate parameters closely tied to research questions.

However, there are relatively few applied studies that implement such procedures.

Reparameterization

Why has reparameterization failed to catch on among social scientists? Possible reasons:

- Reparameterization has often been demonstrated in the context of a single functional form in isolation.
  - May not be clear to potential users how the same procedure could be followed for a variety of functional forms.
- When it has appeared in prior literature, reparameterization has typically not been the primary focus of the article.
- There have not been enough linkages with applied topics to motivate substantive researchers to go the extra length vis-à-vis using unconventional model specifications.

There is a need for a general explanatory framework for reparameterizing models, the goals and outcome of which are closely tied to substantive questions.

Goals of this talk

We describe a general approach for obtaining interpretable reparameterizations of LGMs. In broad strokes, it involves:

1. (**Re)parameterizing** the target function to contain substantively important parameters
2. **Linearizing** the target function to render it specifiable in SEM software
3. **Specifying** the model using the structured latent curve approach
4. **Estimating** model parameters

We start by describing this framework *conceptually*, and then illustrate the details in the context of the three motivating examples.

Throughout, we highlight the generality of the approach and the new substantively relevant information that can be obtained using it.

1.

(Re)parameterization of the
target function to contain substantively
important parameters

Step 1: Reparameterization

Reparameterization proceeds in substeps:

- **1a** Begin with a model expression.
- **1b** Decide what aspect(s) of change we wish to quantify.
- **1c** Determine how that aspect of change could be expressed in terms of existing model parameters (this often requires basic calculus). Derive an expression for that aspect of change.
- **1d** Solve that expression in terms of existing parameters and plug the expression into the original model. This is the reparameterized model.

2.

Linearization of the target function to render it specifiable in SEM software

Step 2: Linearization

In many cases reparameterization will result in an **intrinsically nonlinear** function.

For instance, some parameters may enter the model embedded in reciprocals, radicals, trigonometric terms, exponents, or logarithms.

Step 2: Linearization

This intrinsic nonlinearity poses a practical problem for us because SEM is a fundamentally **linear** framework.

Because SEM requires a model to be linear, we may need to “linearize” it to enable estimating the model in SEM software.

Step 2: Linearization

To linearize the target function, we approximate the target function with a sum, consisting of:

- the target function (evaluated at the parameter estimates)
  +
- parameters of the target function × partial derivatives w.r.t. each parameter

This approximation is called a *Taylor series approximation.*

This approximation has long been used in fitting nonlinear regression and nonlinear mixed models (Davidian & Giltinan, 1995; Hand & Crowder, 1996).

It has only more recently been used in nonlinear SEM.

3.

Specification of the model using the structured latent curve approach

Step 3: Specification using structured latent curves

Following Step 2, the function will be in a linear form, but not necessarily in *the particular linear form* expected by SEM software.

SEM software expects that measured variables are an additive linear combination of coefficients × predictors, plus error.

Step 3: Specification using structured latent curves

As a third step, we employ the principles of **structured latent curve** (SLC; Browne, 1993; Browne & du Toit, 1991) modeling to rearrange the linearized function in order to specify it in a way that SEM software understands.

This entails:

3a Treating the partial derivatives as factor loadings

- This is done using nonlinear constraints in a SEM program

3b Treating random coefficients as latent variables

- Almost any growth parameter can be treated as a random coefficient in this framework.

4.

Estimation of the model parameters

Step 4: Estimation

Once the model is specified, it can be fit using SEM software.

The chosen software must be capable of imposing nonlinear constraints:

- LISREL
- Mplus
- PROC CALIS
- Mx
- OpenMx

It is important to note that the framework described here:

- accommodates MAR missing data (using FIML) and
- can be modified to accommodate individually-varying occasions of measurement.

Summary so far

We covered in conceptual terms 4 steps that can be taken to proceed from a conventional LGM with less interpretable parameters → reparameterized LGM with more interpretable parameters.

1. **(Re)parameterize** the model so that parameters / random coefficients have direct substantive interpretations
2. **Linearize** the model using a Taylor series so that it can be fit using SEM software
3. **Specify** the model as a structured latent curve model
4. **Estimate** the model parameters using SEM software

Now that the groundwork has been laid, we present concrete details in the context of our 3 motivating examples.

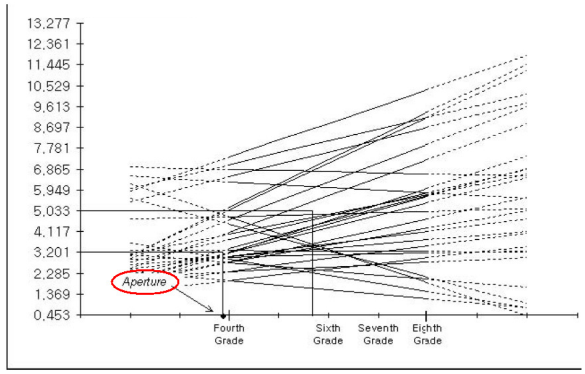
Application of the general framework
to the 3 motivating examples

Motivating example from clinical psychology:
Delinquent affiliation

Motivating example from clinical psychology: Delinquent affiliation

Earlier we mentioned that clinicians are often interested in tracking the degree of children's affiliation with delinquent peers.

In particular, researchers may wish to locate the point in time where children are the most similar to each other in affiliation with delinquents, before their trajectories begin to diverge.

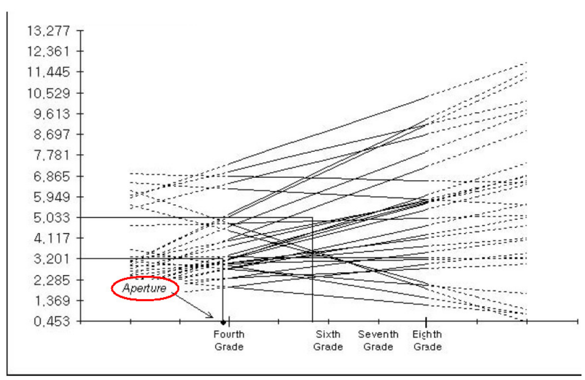


Data are from Stoolmiller (1994).
Figure is from Hancock and Choi (2006).

Motivating example from clinical psychology: Delinquent affiliation

Locating this point would help clinicians properly time interventions to delay or prevent negative behaviors that tend to spread through peer associations —

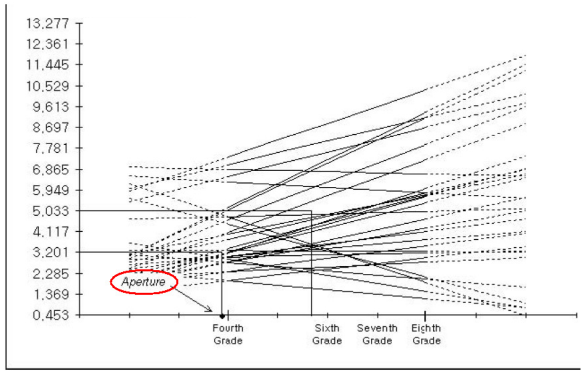
- drug use,
- truancy,
- crime, etc.



Motivating example from clinical psychology: Delinquent affiliation

More generally, this point in time of greatest similarity is called the **aperture** (Hancock & Choi, 2006; Mehta & West, 2000).

The aperture can be directly estimated as a model parameter using the steps described previously.



Motivating example from clinical psychology: Delinquent affiliation

1. (Re)parameterization

1a. Begin with a model expression

The unconditional linear LGM:

$$y_{ij} = \eta_{1j} + \eta_{2j}\left(t_{ij} - t^*\right) + \varepsilon_{ij}$$

- t_{ij} time for person j at occasion i
- t^* time chosen to be the origin
- η_1 latent intercept
- η_2 latent slope

The 6 estimated parameters are 2 means,
3 (co)variances of latent growth factors, and 1 homoscedastic residual variance:

$$\begin{bmatrix} \eta_{1j} \\ \eta_{2j} \end{bmatrix} \sim MVN\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \psi_{11} & \\ \psi_{21} & \psi_{22} \end{bmatrix}\right) \qquad \varepsilon_{ij} \sim N\left(0, \sigma_\varepsilon^2\right)$$

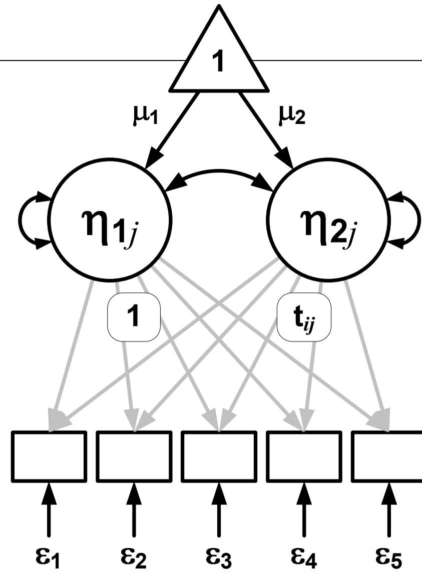
Motivating example from clinical psychology: Delinquent affiliation

1. (Re)parameterization

1a. Begin with a model expression

A path diagram of the model, and one way to represent the model using matrix equations:

$$\mathbf{y}_i = \mathbf{\Lambda\mu} + \mathbf{\Lambda\eta}_i + \mathbf{\varepsilon}_i$$



Motivating example from clinical psychology: Delinquent affiliation

1. (Re)parameterization

1b. Decide what aspect(s) of change we wish to quantify

In this example, we would like to estimate the aperture — the value of *time* at which children are the most similar to one another in terms of affiliation with delinquent peers.

The aperture occurs where the *model-implied variance of y* is smallest.

Motivating example from clinical psychology: Delinquent affiliation

1. (Re)parameterization

1c. Express that aspect of change as a function of model parameters

First, express the model-implied variance of y as a function of model parameters:

$$\sigma_y^2 = \mathrm{var}\left(\eta_{1j} + \eta_{2j}\left(t_{ij} - t^*\right) + \varepsilon_{ij}\right)$$

$$= \mathrm{var}\left(\eta_{1j}\right) + 2\left(t_{ij} - t^*\right)\mathrm{cov}\left(\eta_{1j}, \eta_{2j}\right) + \left(t_{ij} - t^*\right)^2 \mathrm{var}\left(\eta_{2j}\right) + \mathrm{var}\left(\varepsilon_{ij}\right)$$

Motivating example from clinical psychology: Delinquent affiliation

1. (Re)parameterization

1c. Express that aspect of change as a function of model parameters

Then find where the model-implied variance of y is smallest:

$$\frac{\partial \sigma_y^2}{\partial\left(t_{ij} - t^*\right)} = 2\,\mathrm{cov}\left(\eta_{1j}, \eta_{2j}\right) + 2\left(t_{ij} - t^*\right)\mathrm{var}\left(\eta_{2j}\right)$$

Calculate its first derivative with respect to $t_{ij} - t^*$.

$$0 = \psi_{21} + \left(t_{ij} - t^*\right)\psi_{22}$$

Set it = 0.

$$\left(t_{ij} - \eta_a\right) = -\frac{\psi_{21}}{\psi_{22}}$$

The value of t^* where variance is minimized is η_a, the *aperture.*

Motivating example from clinical psychology: Delinquent affiliation

1. (Re)parameterization

1d. Solve the expression in terms of existing parameters and substitute

Therefore, the reparameterized model is:

$$y_{ij} = \eta_{1j} + \eta_{2j}\left(t_{ij} - \eta_a\right) + \varepsilon_{ij}$$

The aperture is now a model parameter.

With intercept/slope/aperture variances and covariances:

$$\begin{bmatrix} \eta_{1j} \\ \eta_{2j} \\ \eta_a \end{bmatrix} \sim MVN\left(\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_a \end{bmatrix}, \begin{bmatrix} \psi_{11} & & \\ 0 & \psi_{22} & \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$\varepsilon_{ij} \sim N\left(0, \sigma_\varepsilon^2\right)$$

If we center at aperture, $\psi_{21} = 0$, so we constrain it = 0.

Aperture doesn't vary.

So, still estimating only 6 parameters; as before.

Motivating example from clinical psychology: Delinquent affiliation

2. Linearization

A first-order Taylor series approximation (k = number of growth parameters):

$$\tilde{y} \approx y\big|_{\mu} + \sum_p \frac{\partial y}{\partial \eta_{pj}}\bigg|_{\mu_p} \left(\eta_{pj} - \mu_p\right)$$

target function evaluated at the parameter estimates

first partial derivatives of y with respect to each coefficient, evaluated at the coefficient means

parameters of target function

Motivating example from clinical psychology: Delinquent affiliation

2. Linearization

$$\tilde{y} \approx y\Big|_{\mu} + \sum_{p} \frac{\partial y}{\partial \eta_{pj}}\Big|_{\mu_p} \left(\eta_{pj} - \mu_p\right)$$

first partial derivatives

$$\frac{\partial y}{\partial \eta_{1j}} = 1 \qquad \frac{\partial y}{\partial \eta_{2j}} = \left(t_{ij} - \mu_a\right) \qquad \frac{\partial y}{\partial \eta_a} = -\mu_2$$

We will use these shortly.

Motivating example from clinical psychology: Delinquent affiliation

3. Specification

Is the linearized function recognizable by SEM software?

SEM software expects the model to look like this:

$$\mathbf{y}_i = \mathbf{\Lambda\mu} + \mathbf{\Lambda\eta}_i + \boldsymbol{\varepsilon}_i$$

Motivating example from clinical psychology: Delinquent affiliation

3. Specification

It may not be obvious, but our linearized approximation (below) adheres to this form.

Following the principles of structured latent curve modeling:

$$\tilde{y} \approx y\big|_{\mu} + \sum_{p} \frac{\partial y}{\partial \eta_{pj}}\bigg|_{\mu_p} \left(\eta_{pj} - \mu_p\right)$$

$$\mathbf{y}_j = \boldsymbol{\Lambda\mu} + \boldsymbol{\Lambda\eta}_j + \boldsymbol{\varepsilon}_j$$

More specifically,...

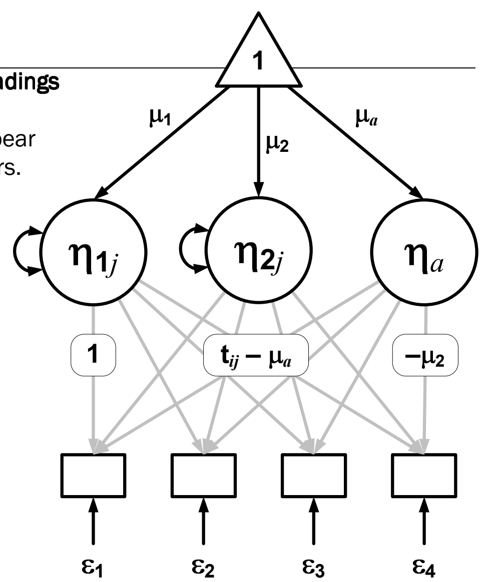
Motivating example from clinical psychology: Delinquent affiliation

3. Specification

3a. Treat the partial derivatives as factor loadings

The derivatives solved for in Step 2 now appear as loadings on their respective growth factors.

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & (0-\mu_a) & -\mu_2 \\ 1 & (1-\mu_a) & -\mu_2 \\ \cdots & \cdots & \cdots \\ 1 & (2-\mu_a) & -\mu_2 \end{bmatrix}$$



Every aspect of change is represented as a factor: intercept, slope, and aperture.

Motivating example from clinical psychology: Delinquent affiliation

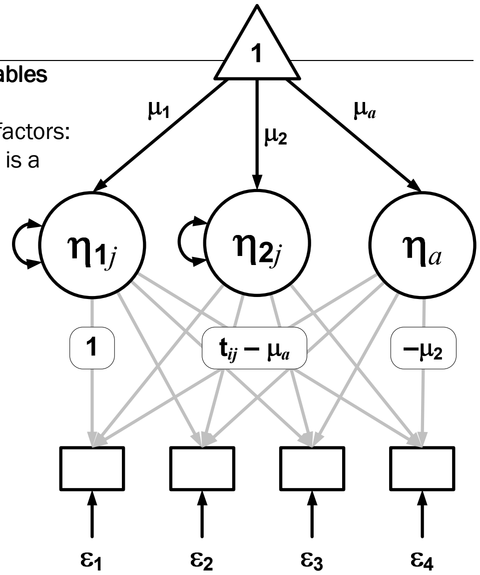
3. Specification

3b. Treat random coefficients as latent variables

All 3 aspects of change are represented as factors: intercept, slope, and aperture. The aperture is a fixed coefficient because it is a summary of level 2 characteristics.

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_a)'$$

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_{11} & & \\ 0 & \psi_{22} & \\ 0 & 0 & 0 \end{bmatrix}$$



CILVR 2010 39

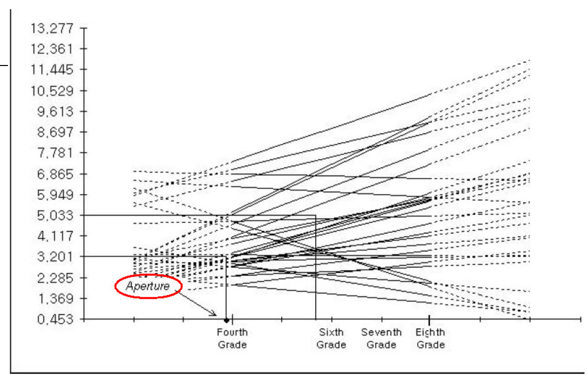
Motivating example from clinical psychology: Delinquent affiliation

4. Estimation

Estimation in LISREL yielded:

$$\boldsymbol{\mu} = \begin{bmatrix} 3.20 \\ .23 \\ 3.63 \end{bmatrix}$$

3.20 ← intercept
.23 ← slope
3.63 ← aperture



$$\boldsymbol{\Psi} = \begin{bmatrix} 5.68 & & \\ 0 & .30 & \\ 0 & 0 & 0 \end{bmatrix}$$

5.68 ← intercept variance at the aperture (minimum variance)
.30 ← slope variance

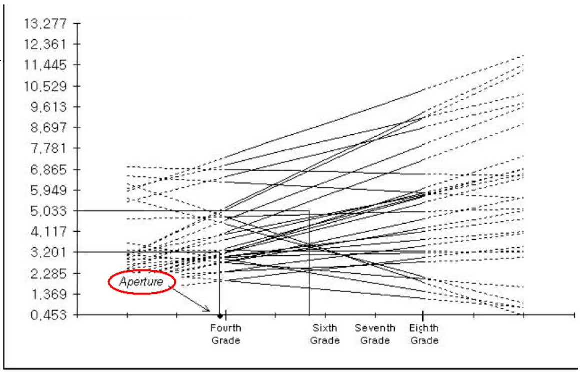
CILVR 2010 40

Motivating example from clinical psychology: Delinquent affiliation

4. Estimation

Estimation in LISREL yielded:

$$\boldsymbol{\mu} = \begin{bmatrix} 3.20 \\ .23 \\ 3.63 \end{bmatrix} \begin{matrix} \leftarrow \text{intercept} \\ \leftarrow \text{slope} \\ \leftarrow \text{aperture} \end{matrix}$$



Conclusion: Children are most similar in their affiliation with delinquents just prior to 4th grade.

Therefore, 3rd grade might be a good time to apply an early intervention program designed to target the prevention of drug use and other negative behaviors that are spread through peer associations.

Motivating example from public health:
Phosphate rebound

Motivating example from public health: Phosphate rebound

Earlier we mentioned that public health researchers are interested in tracking rates of absorption and elimination of chemicals introduced to the body — and individual differences in those rates.

For example:

- Eating carbohydrates signals liver to gather phosphates from blood for digestion
- Phosphates are depleted from blood → diverted to liver → then returned to blood once digestion is completed
- High liver phosphate levels signal satiety

- Obese persons' livers:
  - have low baseline phosphates
  - take longer to uptake phosphates
  - do not uptake as much phosphate

How much food — and how much time — it takes for an individual's liver to signal satiety is related to weight problems. (Obeid et al., 2010).

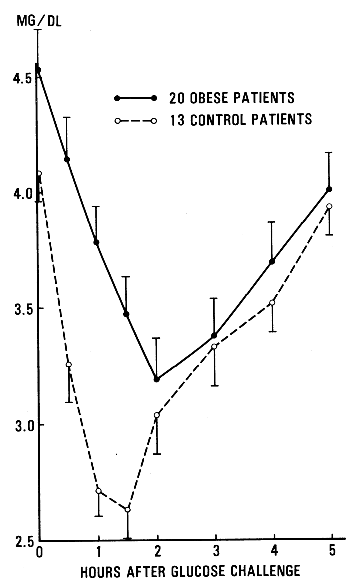
Motivating example from public health: Phosphate rebound

An appropriate model is the **segmented linear spline** (2 phases).

The point where phosphate decline (phase 1) turns into phosphate recovery (phase 2) is called the **knot**, **joint**, or **transition point**.

Investigating interindividual variability in this knot, and obesity's role in predicting it, is necessary but challenging.

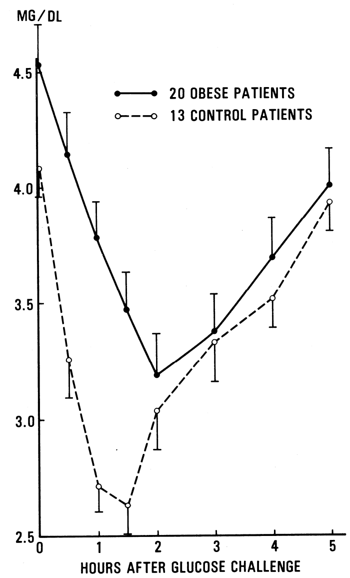
Data and plot from Zerbe (1979).



Motivating example from public health: Phosphate rebound

The knot is not expected to be the same for all people.

Using multilevel modeling (MLM), it has already been shown that knots can be treated as fixed quantities, estimated parameters, or random coefficients (Cudeck & Klebe, 2002; Wang & McArdle, 2008).

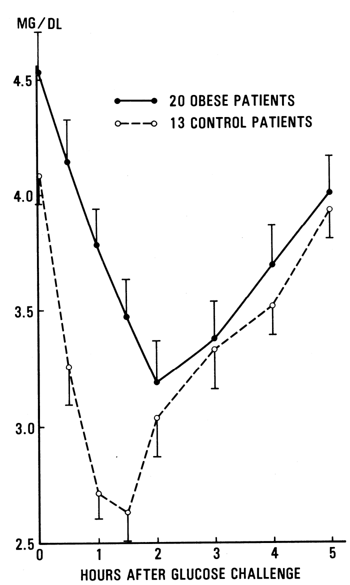


Motivating example from public health: Phosphate rebound

In the SEM context, knots have been treated as fixed quantities (Bollen & Curran, 2006; Flora, 2008) or estimated parameters (Harring, Cudeck, & du Toit, 2006) — not as random coefficients.

It would be beneficial to treat knots as randomly varying across subjects to more accurately mirror individual differences in phosphate rebound.

Doing so in SEM makes available all the advantages of latent variable modeling.



Motivating example from public health: Phosphate rebound

1. (Re)parameterization

1a. Begin with a model expression

Traditional expression of a segmented linear spline:

$$y_{ij} = \begin{cases} \eta_{1j} + \eta_{2j} t_{ij} + \varepsilon_{ij} & t_{ij} \le \eta_{\kappa j} \\ \eta_{3j} + \eta_{4j} t_{ij} + \varepsilon_{ij} & t_{ij} > \eta_{\kappa j} \end{cases}$$

η_{1j} and η_{2j} are the intercept and slope for the first segment for person j.

η_{3j} and η_{4j} are the intercept and slope for the second segment for person j.

$\eta_{\kappa j}$ is the knot (time where the phases shift) for person j.

The segments are assumed to join at the knot, so there are effectively 4 free parameters in the target function.

Motivating example from public health: Phosphate rebound

1. (Re)parameterization

1b. Decide what aspect(s) of change we wish to quantify

We would like to estimate the knot as a parameter <u>or</u> treat it as a random coefficient.

$$y_{ij} = \begin{cases} \eta_{1j} + \eta_{2j} t_{ij} + \varepsilon_{ij} & t_{ij} \le \eta_{\kappa j} \\ \eta_{3j} + \eta_{4j} t_{ij} + \varepsilon_{ij} & t_{ij} > \eta_{\kappa j} \end{cases}$$

However, it is not possible to specify this parameterization in SEM directly in a way that permits treating the knot as an estimated parameter or random coefficient.

Motivating example from public health: Phosphate rebound

1. (Re)parameterization

1c. Express that aspect of change as a function of model parameters

We use the reparameterization from Harring et. al.'s (2006) fixed-but-estimated knot linear spline LGM:

$$y_{ij} = \omega_{1j} + \omega_{2j} t_{ij} + \omega_{3j} \sqrt{\left(t_{ij} - \eta_{\kappa j}\right)^2} + \varepsilon_{ij}$$

The three ω coefficients are functions of the original growth coefficients $\eta_{1j} - \eta_{4j}$. They still represent aspects of change (average intercept across segments, average slope across segments, and half the difference in slopes).

But we are not concerned with them here.

The important point here is that $\eta_{\kappa j}$ in the reparameterized model bears the *same interpretation* as it does in the original model: it is the knot or transition point.

That is, $\eta_{\kappa j}$ "survived" the reparameterization intact.

Motivating example from public health: Phosphate rebound

2. Linearization

Derivatives of the reparameterized target function with respect to the 4 aspects of change are:

$$\frac{\partial y}{\partial \boldsymbol{\eta}} = \begin{bmatrix} 1 \\ t_{ij} \\ \sqrt{\left(t_{ij} - \mu_\kappa\right)^2} \\ \dfrac{\mu_3\left(\mu_\kappa - t_{ij}\right)}{\sqrt{\left(t_{ij} - \mu_\kappa\right)^2}} \end{bmatrix}$$

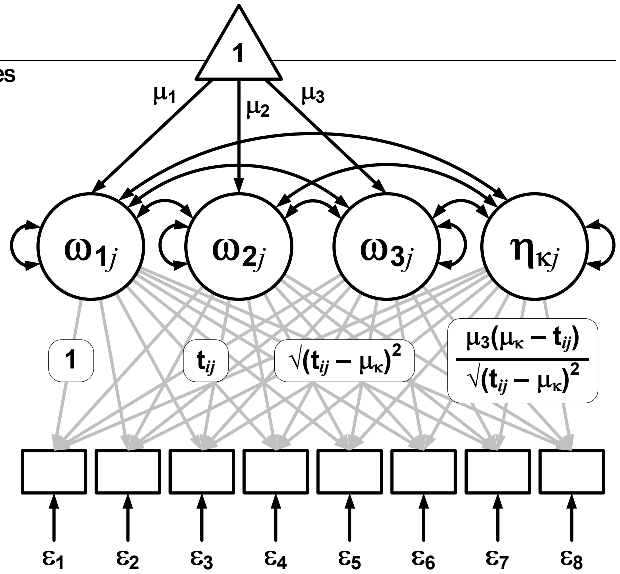
Recall that the first three growth factors are not our main concern here. Our main concern is the fourth: the knot factor.

Motivating example from public health: Phosphate rebound

3. Specification

3a. Treat the partial derivatives as factor loadings

The derivatives with respect to each growth factor are placed in that factor's column in $\boldsymbol{\Lambda}$.



Motivating example from public health: Phosphate rebound

3. Specification

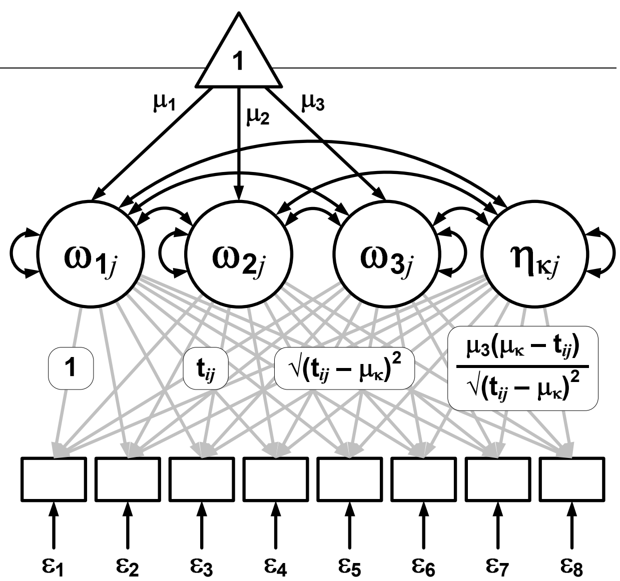
3b. Treat random coefficients as latent variables

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, 0)'$$

We set = 0 to ensure $\boldsymbol{\Lambda\mu}$ equals target function

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_{\omega_1} & & & \\ \psi_{\omega_2,\omega_1} & \psi_{\omega_2} & & \\ \psi_{\omega_3,\omega_1} & \psi_{\omega_3,\omega_2} & \psi_{\omega_3} & \\ \psi_{\kappa,\omega_1} & \psi_{\kappa,\omega_2} & \psi_{\kappa,\omega_3} & \psi_{\kappa} \end{bmatrix}$$

All 4 aspects of change are being treated as random coefficients.



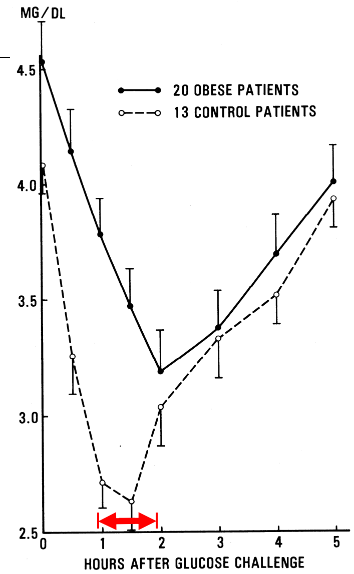
Motivating example from public health: Phosphate rebound

4. Estimation

Regressing the growth coefficients on *obesity* and fitting the SLC model in Mplus yields an obesity → $\eta_{\kappa j}$ effect of **1.013**, with 95% bootstrap confidence interval **{.623, 1.645}**.

That is, > *1 hour* elapses between control vs. obese samples' phosphate rebound points.

This difference is statistically significant.



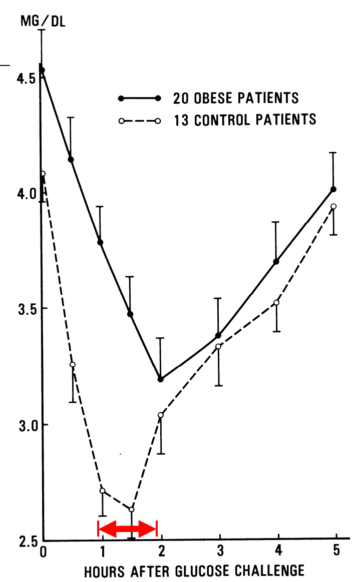
CILVR 2010 53

Motivating example from public health: Phosphate rebound

4. Estimation

Thus, using the reparameterized model, we are now able to predict individual differences in how the body signals satiety after eating.

Accurate prediction of the timing of these signals could help researchers design interventions to manipulate them.



CILVR 2010 54

Motivating example from sociology:
Infant growth

Motivating example from sociology: Infant growth

As stated earlier, sociologists are concerned with tracking infant development in countries that are susceptible to child malnutrition.

Recent studies sponsored by UNICEF (e.g., 1997, 2008) found that African and Asian children are particularly likely to suffer from stunted growth due to malnutrition, and stunting can begin in the womb:

- 49% of Bangladeshi children
- 28% of Iraqi children
- 19% of Somali children
- 26% of children in east Asia (discounting China)
- 35% of East Timorese children

These studies aim to identify determinants of optimal and suboptimal infant growth.

Motivating example from sociology: Infant growth

In order to identify these determinants, we need a model that accurately reflects growth in infant weight.

However, we want to not only describe individual differences in growth trends, but also predict these individual differences — at any desired age — with environmental variables like breastfeeding/bottle-feeding, urban/rural status, etc.

Motivating example from sociology: Infant growth

The Cebu Longitudinal Health and Nutrition Survey includes weight data for Filipino infants every 2 months from 0-24 months (N = 2,632).

The aims of the survey included:

- tracking inter-infant differences in weight gain at various ages.
- determining the extent to which cumulative breastfeeding affects weight at any given age.

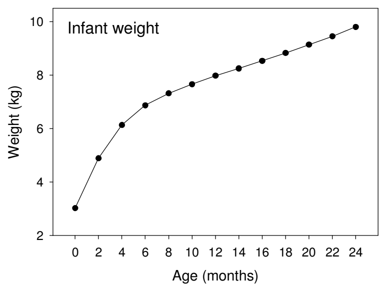
Motivating example from sociology: Infant growth

1. (Re)parameterization

1a. Begin with a model expression

First, we have to choose a plausible function to model growth. The **Jenss-Bayley model** was designed specifically to model infant growth (Jenss & Bayley, 1937).

$$y_{ij} = \eta_{1j} + \eta_{2j} t_{ij} - \exp\left(\eta_{3j} + \eta_{4j} t_{ij}\right) + \varepsilon_{ij}$$



It combines exponential and linear growth. This model is ideal for these data because:

(a) *early* biological growth often does follow an exponential process (where growth acts to limit further growth), but

(b) the asymptote is not horizontal.

Motivating example from sociology: Infant growth

1. (Re)parameterization

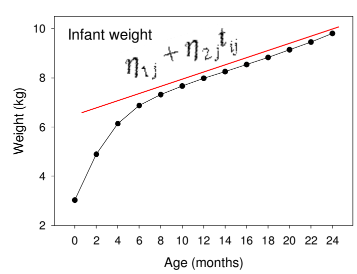
1a. Begin with a model expression

The Jenss-Bayley model:

$$y_{ij} = \eta_{1j} + \eta_{2j} t_{ij} - \exp\left(\eta_{3j} + \eta_{4j} t_{ij}\right) + \varepsilon_{ij}$$

the linear part forms the asymptote

the exponential part determines growth toward the asymptote



The exponential growth component merges seamlessly with the linear growth component as time progresses.

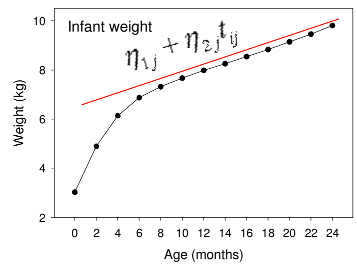
Motivating example from sociology: Infant growth

1. (Re)parameterization

1a. Begin with a model expression

The Jenss-Bayley model:

$$y_{ij} = \eta_{1j} + \eta_{2j} t_{ij} - \exp\left(\eta_{3j} + \eta_{4j} t_{ij}\right) + \varepsilon_{ij}$$



η_{1j} is the intercept of the linear asymptote

η_{2j} is the slope of the linear asymptote

$\exp\left(\eta_{3j}\right)$ is the vertical distance between the actual intercept and the linear asymptote's intercept

$\exp\left(\eta_{4j}\right)$ is the ratio of the acceleration of growth at age t to that at age $t - 1$.

Motivating example from sociology: Infant growth

1. (Re)parameterization

1b. Decide what aspect(s) of change we wish to quantify

We want a parameterization that lets us predict individual differences in infant weight at *any given point* in time, 0 – 24 months (t^*).

We need to reparameterize the Jenss-Bayley function in such a way that model-implied weight at time t^* (i.e., η^*) is a random effect.

You might think we could simply

- recenter age at a variety of different points, and
- reestimate the model each time to capture individual differences at various ages, as we might in a linear or polynomial growth curve.

But when we use intrinsically nonlinear functions (as is required for charting physical growth), it is not so straightforward.

Motivating example from sociology: Infant growth

1. (Re)parameterization

1c. Express that aspect of change as a function of model parameters

Model-implied weight at an arbitrary age t^* is described by the target function with t^* substituted for t_{ij}:

$$\eta^* = \eta_1 + \eta_2 t^* - \exp\left(\eta_3 + \eta_4 t^*\right)$$

Here, the model-implied weight at a specific time t^* — our new parameter η^* — is expressed as a function of other model parameters.

Motivating example from sociology: Infant growth

1. (Re)parameterization

1d. Solve the expression in terms of existing parameters and substitute

$$\eta^* = \eta_1 + \eta_2 t^* - \exp\left(\eta_3 + \eta_4 t^*\right)$$

We decided to solve this expression for the linear intercept η_1 and substitute the result back into the Jenss-Bayley model, yielding:

$$y_{ij} = \eta_j^* + \eta_{2j}\left(t_{ij} - t^*\right) + \exp\left(\eta_{3j} + \eta_{4j} t^*\right) - \exp\left(\eta_{3j} + \eta_{4j} t_{ij}\right) + \varepsilon_{ij}$$

This is the reparameterized model.

Motivating example from sociology: Infant growth

2. Linearization

The derivatives of this reparameterized model with respect to each growth factor are:

$$\frac{\partial y}{\partial \boldsymbol{\eta}} = \begin{bmatrix} 1 \\ \left(t_{ij} - t^*\right) \\ \exp\left(\mu_3 + \mu_4 t^*\right) - \exp\left(\mu_3 + \mu_4 t_{ij}\right) \\ t^* \exp\left(\mu_3 + \mu_4 t^*\right) - t_{ij} \exp\left(\mu_3 + \mu_4 t_{ij}\right) \end{bmatrix}$$

Motivating example from sociology: Infant growth

3. Specification

3a. Treat the partial derivatives as factor loadings

Putting the derivatives with respect to each of the 4 growth factors in their respective columns of $\boldsymbol{\Lambda}$, we have:

$$\boldsymbol{\Lambda} = \left[\begin{array}{c|c|c|c} 1 & 0 - t^* & \exp\left(\mu_3 + \mu_4 t^*\right) - \exp\left(\mu_3\right) & t^* \exp\left(\mu_3 + \mu_4 t^*\right) \\ 1 & 2 - t^* & \exp\left(\mu_3 + \mu_4 t^*\right) - \exp\left(\mu_3 + 2\mu_4\right) & t^* \exp\left(\mu_3 + \mu_4 t^*\right) - (2)\exp\left(\mu_3 + 2\mu_4\right) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 24 - t^* & \exp\left(\mu_3 + \mu_4 t^*\right) - \exp\left(\mu_3 + 24\mu_4\right) & t^* \exp\left(\mu_3 + \mu_4 t^*\right) - (24)\exp\left(\mu_3 + 24\mu_4\right) \end{array}\right]$$

Motivating example from sociology: Infant growth

3. Specification

3b. Treat random coefficients as latent variables

All four parameters can be treated as random coefficients (latent variables with estimated variances and covariances) using SLC principles.

Later we will also include infant-level predictors (e.g., cumulative breastfeeding) of all these random coefficients, and several desired ages.

Predicted wt. at time t^*

$$\boldsymbol{\mu} = (\mu^*, \mu_2, \mu_3, 0)'$$

Fixed to 0 to make $\boldsymbol{\Lambda\mu}$ = the target function. The mean for rate parameter is estimated inside $\boldsymbol{\Lambda}$.

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_{\eta^*} & & & \\ \psi_{\eta_2,\eta^*} & \psi_{\eta_2} & & \\ \psi_{\eta_3,\eta^*} & \psi_{\eta_3,\eta_2} & \psi_{\eta_3} & \\ \psi_{\eta_4,\eta^*} & \psi_{\eta_4,\eta_2} & \psi_{\eta_4,\eta_3} & \psi_{\eta_4} \end{bmatrix}$$

Motivating example from sociology: Infant growth

4. Estimation

We estimate the (unconditional) reparameterized model at several values of t^*:

$$y_{ij} = \eta_j^* + \eta_{2j}(t_{ij} - t^*) + \exp(\eta_{3j} + \eta_{4j} t^*) - \exp(\eta_{3j} + \eta_{4j} t_{ij}) + \varepsilon_{ij}$$

and we plot the model-implied mean weight, μ^* at time t^*, along with a 95% interval based on the variance of the model-implied weight η_j^*.

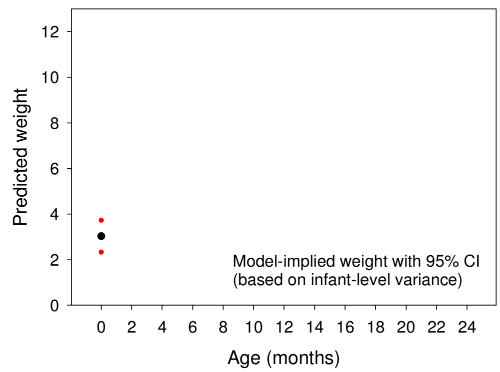
Motivating example from sociology: Infant growth

4. Estimation

We estimate the (unconditional) reparameterized model at several values of t^*:

$$y_{ij} = \eta_j^* + \eta_{2j}(t_{ij} - 0) + \exp(\eta_{3j} + \eta_{4j}0) - \exp(\eta_{3j} + \eta_{4j}t_{ij}) + \varepsilon_{ij}$$

Model-implied weight, μ^*
And the interval
$\mu^* \pm 1.96\sqrt{\psi_{\eta^*}}$
where t^* = **0**.



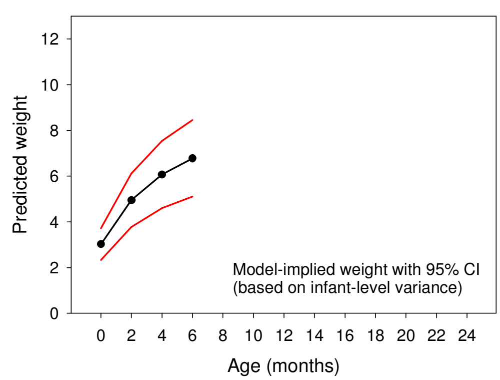
Motivating example from sociology: Infant growth

4. Estimation

We estimate the (unconditional) reparameterized model at several values of t^*:

$$y_{ij} = \eta_j^* + \eta_{2j}(t_{ij} - 6) + \exp(\eta_{3j} + \eta_{4j}6) - \exp(\eta_{3j} + \eta_{4j}t_{ij}) + \varepsilon_{ij}$$

Model-implied weight, μ^*
And the interval
$\mu^* \pm 1.96\sqrt{\psi_{\eta^*}}$
where t^* = **6**.



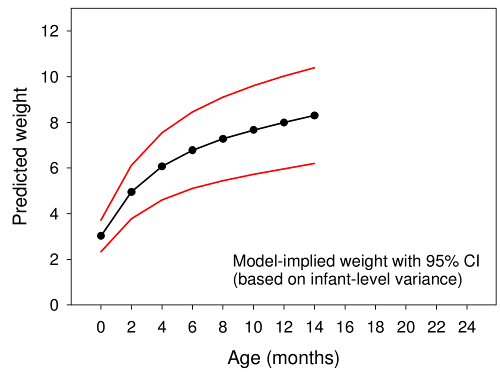
Motivating example from sociology: Infant growth

4. Estimation

We estimate the (unconditional) reparameterized model at several values of $t*$:

$$y_{ij} = \eta_j^* + \eta_{2j}\left(t_{ij} - 14\right) + \exp\left(\eta_{3j} + \eta_{4j}14\right) - \exp\left(\eta_{3j} + \eta_{4j}t_{ij}\right) + \varepsilon_{ij}$$

Model-implied weight, μ^*
And the interval
$\mu^* \pm 1.96\sqrt{\psi_{\eta^*}}$
where t^* = **14**.



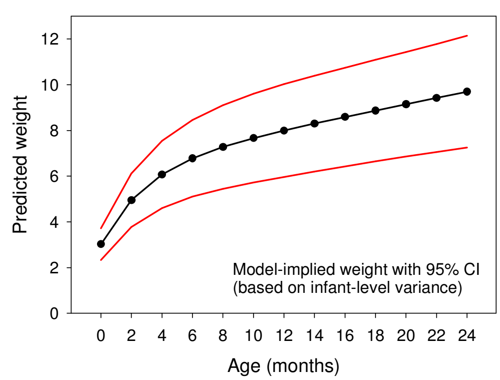
Motivating example from sociology: Infant growth

4. Estimation

We estimate the (unconditional) reparameterized model at several values of $t*$:

$$y_{ij} = \eta_j^* + \eta_{2j}\left(t_{ij} - 24\right) + \exp\left(\eta_{3j} + \eta_{4j}24\right) - \exp\left(\eta_{3j} + \eta_{4j}t_{ij}\right) + \varepsilon_{ij}$$

Model-implied weight, μ^*
And the interval
$\mu^* \pm 1.96\sqrt{\psi_{\eta^*}}$
where t^* = **24**.



Motivating example from sociology: Infant growth

4. Estimation

Overall…

μ^* followed the Jenss-Bayley function over time

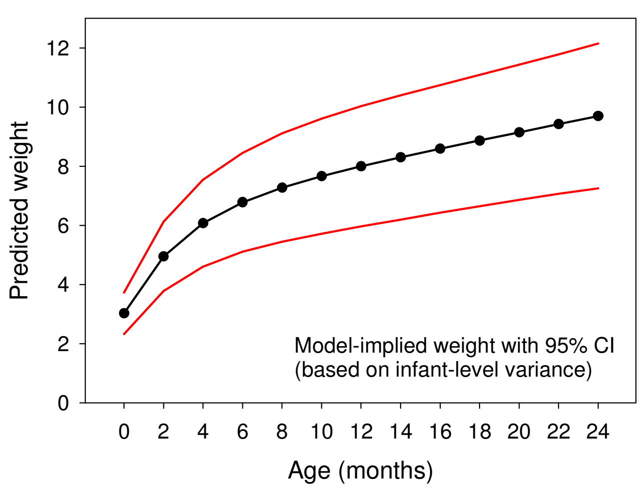
$\mu_2 = .137$ (slope of the linear asymptote)

$\exp(\mu_3) = 3.047$ (distance b/t actual intercept and linear asymptote's intercept)

$\exp(\mu_4) = .716$ (ratio of the acceleration of growth at age t to that at age t – 1)

Motivating example from sociology: Infant growth

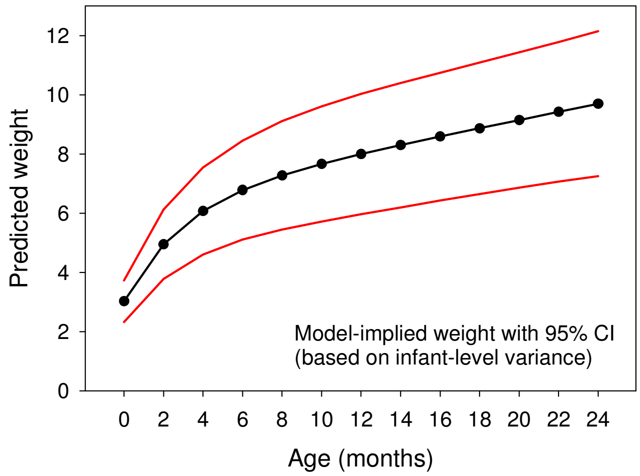
4. Estimation



Shifting the value of t^* and running the unconditional model 13 times gives us this plot of conditional means.

Motivating example from sociology: Infant growth

4. Estimation



Each dot is derived from a different reparameterized model — one in which time is "centered" at that age (t^*).

The next step was to use breastfeeding to *predict* inter-infant variability in weight at each age.

Motivating example from sociology: Infant growth

4. Estimation

Our primary interest — and the reason we were obliged to reparameterize the model in the first place — is in predicting individual differences in weight at any given age by cumulative breastfeeding *up to that point*.

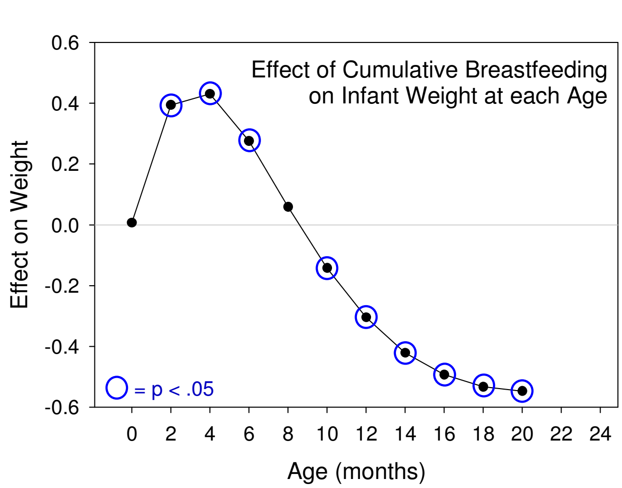
At every occasion of measurement, mothers were asked whether they had breastfed the previous day (0, 1).

Cumulative breastfeeding was defined as the average of all breastfeeding responses up to a given point in time.

- We introduced cumulative breastfeeding as an infant-level predictor of all four random coefficients.
- Interest focused on its effect on infant weight.

Motivating example from sociology: Infant growth

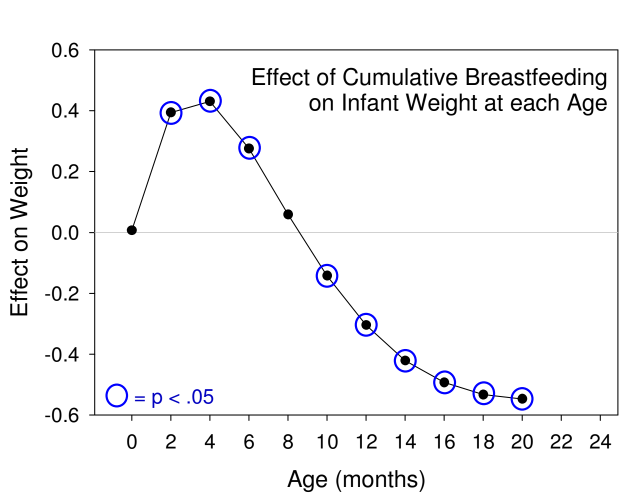
4. Estimation



Each dot represents the slope of weight regressed on average breastfeeding activity *up until that age*.

Motivating example from sociology: Infant growth

4. Estimation

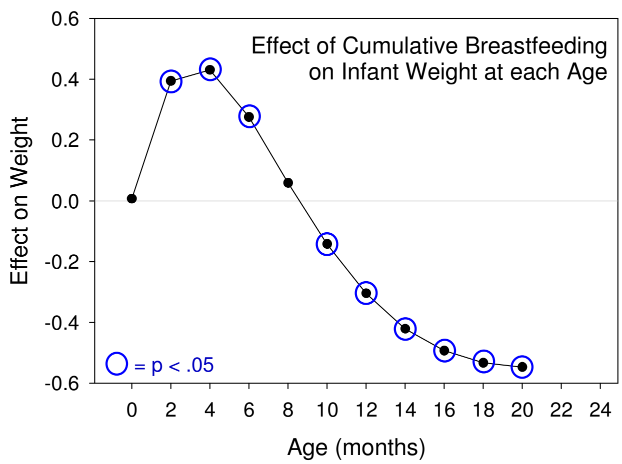


This pattern of effects tells us something useful — the effect of breastfeeding on infant weight is *positive* in the early months, but *negative* at 10 months and beyond.

Steep drops occur by 6 months — incidentally when the Am. Acad. of Ped., WHO, UNICEF, etc. recommend beginning solid food.

Motivating example from sociology: Infant growth

4. Estimation



Follow-up analyses (not presented) show that this effect is not simply attributable to rural/urban differences in breastfeeding practices.

Summary

We have provided a framework for reparameterizing LGMs to yield new parameters, their SEs, and CIs to answer important substantive questions.

We presented four steps for using SEM to model trends:

1. **(Re)parameterization** of the target function to contain important new parameters
2. **Linearization** of the target function to render it specifiable in SEM software
3. **Specification** of the model using the structured latent curve approach
4. **Estimation** of model parameters

Summary

We described this approach conceptually, then gave more detailed explanations in the context of:

- A developmental psychology example on affiliation with delinquent peers
- A public health example on phosphate rebound
- A sociological example on infant growth

Discussion

Reparameterization makes the already-flexible SEM even more flexible.

Particularly in longitudinal settings, it permits us to treat virtually any aspect of change as:

- a fixed, known value
- an estimated parameter
- a random coefficient

The latter two options provide a way to investigate whether these aspects of change are predicted / moderated by level-2 predictors.

Discussion

We examined three examples of new parameters one could estimate:

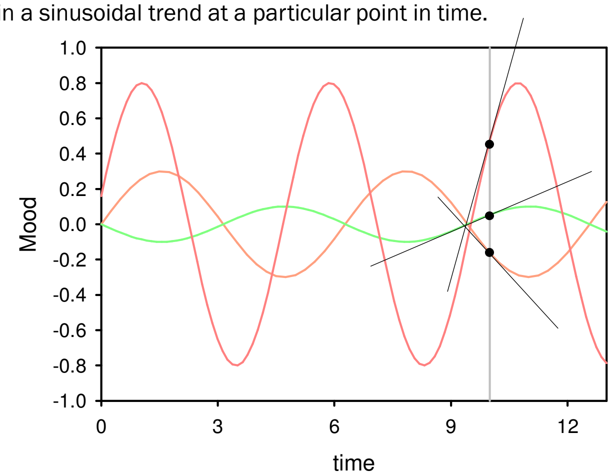
- the aperture point
- knot or transition point
- predicted status at t*

Reparameterization could be treated explicitly as a modeling strategy in graduate courses on longitudinal modeling.

Discussion

Other ideas for new parameters with potential specific uses:

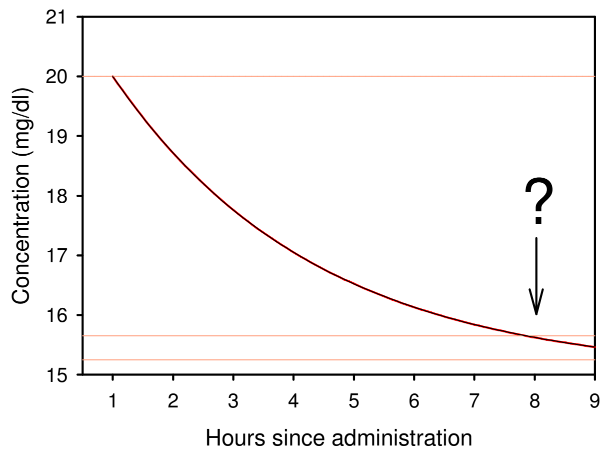
- estimating / predicting individual differences in the instantaneous rate of change in a sinusoidal trend at a particular point in time.



Discussion

Other ideas for new parameters with potential specific uses:

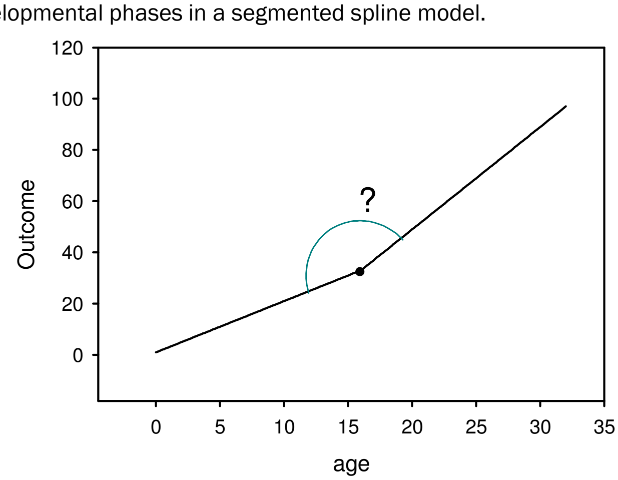
- estimating / predicting the amount of time it takes for 95% of a drug to leave one's system.



Discussion

Other ideas for new parameters with potential specific uses:

- estimating / predicting the degree of smooth continuity between two developmental phases in a segmented spline model.



Thank You

Syntax for analyses available at: http://www.quantpsy.org.